KÄHLER-RICCI SHRINKERS AND FANO FIBRATIONS, II

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Abstract. This paper is the second in a two-part series on Kähler–Ricci shrinkers and Fano fibrations. We introduce the notion of weighted volume for Fano fibrations and establish several of its basic properties. We also formulate a series of conjectures relating Kähler–Ricci shrinkers to Fano fibrations from both algebraic and metric-geometric perspectives.

1. Introduction

A Kähler-Ricci shrinker (i.e., a complete shrinking gradient Kähler-Ricci soliton) consists of the data (X, J, ω, f), where (X, J, ω) is a complete Kähler manifold, and f is a real-valued function on X such that $J\nabla f$ is a holomorphic Killing field and the following equation holds

$$\mathrm{Ric}(\omega) + \sqrt{-1}\partial\bar{\partial} f = \omega. \tag{1.1}$$

The vector field $J\nabla f$ is referred to as the *soliton vector field*. We normalize f so that

$$\Delta_\omega f + f - |\nabla^{1,0} f|^2 = 0. \tag{1.2}$$

The following result was established in our previous paper, the first part of this series. We refer the reader to [SZ26, Section 2] for the definitions of the notions used below.

Theorem 1.1. *A Kähler-Ricci shrinker (X, J, ω, f) naturally defines a polarized Fano fibration. In particular, X is quasi-projective.*

Given Theorem 1.1, then we make precise a general conjecture relating the existence of Kähler-Ricci shrinkers and stability in algebraic geometry.

Conjecture 1.2. *A smooth polarized Fano fibration $(\pi : X \to Y, \xi)$ admits a Kähler-Ricci shrinker, which is unique up to the action of $\mathrm{Aut}(X, \xi)$ (the group of automorphisms of X preserving ξ), if and only if it is K-polystable.*

The notion of K-stability for polarized Fano fibrations will be defined in Section 3. In general, we allow X to have singularities and thus, an appropriate notion of a *singular* Kähler–Ricci shrinker is introduced; see Section 4.1 for a more detailed discussion on this. Conjecture 1.2, together with its singular version Conjecture 4.2, sets a uniform ground for the Yau-Tian-Donaldson type conjectures for Fano type situations studied previously, including positive Kähler-Einstein metrics, Ricci-flat Kähler cones, and compact Kähler-Ricci shrinkers. Along the way, we introduce a local invariant, the *weighted volume* $\mathbb{W}$, for Fano fibrations. This unifies the normalized volume in [Li18] for a klt singularity and the $\tilde{\beta}$-invariant in [HL24] for a Fano variety. Moreover we establish the following local-to-global volume comparison as an algebraic analogue of the Bishop–Gromov volume comparison.

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Theorem 1.3. *Given a Fano fibration germ $(\pi : X \to Y, p)$ and $q \in \pi^{-1}(p)$, we have the following local-to-global comparison for weighted volume*

$$\mathbb{W}(X, q) \geq \mathbb{W}(\pi, p). \tag{1.3}$$

The Yau-Tian-Donaldson type conjecture in general predicts that the existence of canonical Kähler metrics is equivalent to certain stability notions in algebraic geometry. The aim is to relate the question of solving a difficult non-linear partial differential equation to more amenable algebro-geometric numerical conditions.

For Kähler-Einstein metrics on Fano manifolds the Yau-Tian-Donaldson conjecture was proved in [CDS15], which provides an algebro-geometric criterion in terms of K-polystability. The easier direction that the existence of Kähler-Einstein metrics implies K-polystability was previously known by [Ber16] for general Fano varieties. One follow-up research direction focused on extending the result of [CDS15] to Fano varieties. This was established for $\mathbb{Q}$-Gorenstein smoothable Fano varieties by [SSY16]. Based on the variational approach initiated in [BBJ21], it is established in [Li22] the case of general Fano varieties satisfying a stronger notion of reduced uniform K-stability. The algebro-geometric study of K-stability of Fano varieties has seen substantial progress recently, using deep techniques from birational geometry such as those in [HMX14] and [Bir19, Bir21]. In particular, it is proved in [LXZ22] that K-polystability of a Fano variety is equivalent to reduced uniform K-stability, which then completes the algebro-geometric aspect for general Fano varieties. On the differential geometric side however, there are still intriguing questions on the geometric behavior of the singular Kähler-Einstein metrics and linking them to metric geometry (for example RCD spaces); there are important recent progress [GPSS24, Szé24]. For Ricci-flat Kähler cone metrics on Fano cones (or Sasaki-Einstein metrics), the YTD type conjecture, formulated in [CS18], was proved in [CS19] for isolated singularities, and in [Li21] for general Fano cones.

Kähler-Ricci shrinkers are natural generalizations of both positive Kähler-Einstein metrics (where $f = 0$) and of Ricci-flat Kähler cones (where $\mathrm{Ric} = 0$). In the compact setting, it was proved by [TZ02] that the soliton vector field is a holomorphic invariant of the Fano manifold which minimizes a strictly convex functional and it was proved in [TZ00, TZ02] that Kähler–Ricci shrinkers when exist are unique up to automorphisms. A YTD type conjecture for compact Kähler-Ricci shrinkers is known to hold due to the work of [BWN14] and [DS16] (see also [CSW18]). Generalizations to singular Fano varieties are proved by [BWN14, HL23, BLXZ23].

In the non-compact setting a systematic study of Kähler-Ricci shrinkers was initiated in [CDS24]. In particular, it was realized, using the Duistermaat–Heckman localization formula in symplectic geometry, that the soliton vector field still can be determined by a weighted volume minimization principle in suitable settings. This reveals the "Fano" aspects of non-compact Kähler-Ricci shrinkers, which behave better than the more extensively studied case of Ricci-flat Kähler metrics. It also allows some partial classification theory of Kähler-Ricci shrinkers. In [CDS24] a preliminary attempt was made to relate the existence of Kähler-Ricci shrinkers to algebro-geometric stability. A complete classification of non-compact Kähler-Ricci shrinkers in complex dimension 2 is known, due to the works in [CDS24, CCD24b, BCCD24, LW23]. Conjecture 1.2 incorporates the YTD type conjecture for all the above special cases.

Another goal of this paper is to describe a picture for understanding singularities of Kähler-Ricci flow space-time in terms of local algebraic geometry of Fano fibrations, extending the local singularity theory in [DS17]. The general scheme is to connect tangent cones of singularities in canonical differential geometric objects with algebraic geometry. Tangent cones,

arising from rescalings and often showing natural symmetries, are fundamental in studying singularities in geometric PDEs. However, their uniqueness, independent of rescaling subsequences, remains a well-known open problem. In [DS17] the uniqueness of tangent cones of non-collapsed singularities of polarized Kähler-Einstein metrics was established by drawing connections with algebraic geometry. More importantly, [DS17] gives an algebro-geometric description of the unique tangent cone through a *2-step degeneration theory* in terms of a valuation canonically defined by the metric. The latter is conjectured in [DS17] to be an algebro-geometric invariant and to lead to a local stability theory for algebraic singularities. [Li18] related this picture to a generalized volume minimization in Sasaki geometry and reformulated this conjecture using more algebro-geometric terminology. The conjecture was confirmed in [LX18, LWX21] and the algebro-geometric aspect, referred to as the stable degeneration conjecture, is further extended and developed by many people (see for example [Blu18, BL21, Xu20, XZ21, XZ22, XZ24] and the references therein).

A parabolic generalization of this picture is the setting of singularities of non-collapsed Kähler-Ricci flow space-time. Natural examples include finite time singularities and ancient solutions of Kähler-Ricci flow and the static space of a singular Ricci-flat Kähler metric. The corresponding tangent flows are exactly Kähler-Ricci shrinkers (with possible singularities). On a Fano manifold, it was discovered by [CSW18] that the long-time behavior of the *normalized* Kähler–Ricci flow, when the initial class is a positive multiple of the anticanonical class, generates new algebro-geometric degenerations for Fano manifolds. In particular, it was proved that any long-time solution to the normalized Ricci flow on a Fano manifold leads to a 2-step degeneration theory (similar to [DS17]) to a unique Kähler-Ricci shrinker on a Fano variety. It was conjectured in [CSW18] that this degeneration is independent of the initial data hence inducing an optimal degeneration which is an algebro-geometric invariant of a Fano manifold. This conjecture was confirmed by [HL24] for smooth Fano manifolds, based on earlier work of [He16] and [DS20]. Further algebro-geometric aspect of the conjecture on general Fano varieties was proved in [BLXZ23].

In this paper we formulate a series of conjectures that would yield the uniqueness of tangent flows of non-collapsed Kähler-Ricci flow space-time, in terms of a 2-step degeneration picture for the *weighted volume* minimization; see Section 4. It is very likely that the deep work of [Bir19, Bir21, Bir22, Bir23] combined with suitable extensions of the previous techniques [Xu20, BLX22, HL23, BLXZ23, LWX21, XZ22] could be used to solve the algebro-geometric aspects of these conjectures. Moreover, we expect the *weighted volume*, as a local invariant for Fano fibrations, to play a role in the study of boundedness questions in birational geometry. We leave these for future study.

Remark. This is Part II of our 2024 arXiv preprint *Kähler–Ricci shrinkers and Fano fibrations* (arXiv:2410.09661). Since then, considerable progress has been made on some of our conjectures. To preserve the historical context, we have retained the original text, apart from making the proposed definitions of the degree functions in Sections 4.4 and 4.5 more precise.

2. Weighted volume of Fano fibration germs

Fix a Fano fibration germ $(\pi : X \to Y, p)$ with $\dim X = n$. Fix a positive integer r such that $L = -rK_X$ is a Cartier divisor. Since π is proper, for any $m \geq 0$, the push-forward $\pi_*\mathcal{O}(mL)$ is a coherent sheaf on Y. Denote by R_m the stalk of $\pi_*\mathcal{O}(mL)$ at p. Then we obtain a graded $\mathcal{O}_{Y,p}$-algebra

$$\mathcal{R} := \bigoplus_{m\geq 0} R_m = \bigoplus_{m\geq 0} (\pi_*\mathcal{O}(mL))_p. \tag{2.1}$$

Consider a real valuation v on the function field $K(X)$ which is centered in $\pi^{-1}(p)$. This means that if $\mathcal{O}_v$ is the valuation ring of v, then there is a unique scheme-theoretic point $c_X(v) \in X$, the *center* of v, such that we have a local inclusion of local rings $\mathcal{O}_{X,c_X(v)} \hookrightarrow \mathcal{O}_v$ and the Zariski irreducible closed set $\overline{\{c_X(v)\}}$ is contained in $\pi^{-1}(p)$. Moreover, in the case when π is birational a valuation on $K(X)$ centered in $\pi^{-1}(p)$ is the same as a valuation on $K(Y)$ centered at p.

The valuation v induces a valuation on $\mathcal{R}$ as follows. Choose a closed point $q \in \overline{\{c_X(v)\}}$ and trivialize L in an affine neighborhood of q in X, then we have injective maps:

$$R_m \hookrightarrow \mathcal{O}_{X,q} \hookrightarrow \mathcal{O}_{X,c_X(v)} \hookrightarrow K(X). \tag{2.2}$$

It is straightforward to show that the induced valuation on $\mathcal{R}$ is independent of the choices. The valuation induces a filtration $\mathcal{F}_v^\bullet$ on the graded algebra $\mathcal{R}$, by setting

$$\mathcal{F}_v^\lambda R_m := \{f \in R_m | v(f) \geq \lambda\}.$$

Then as in [WN12, BC11, BKMS15, BHJ17] we consider

$$\mathrm{DH}_m := \frac{1}{(rm)^n} \sum_{\lambda \geq 0} \dim_{\mathbb{C}}(\mathcal{F}_v^{\lambda rm} R_m / \mathcal{F}_v^{>\lambda rm} R_m) \delta_\lambda. \tag{2.3}$$

Proposition 2.1. *For each $m \geq 1$, DH_m defines a Radon measure on $[0, \infty)$. Moreover, as $m \to \infty$, DH_m converges weakly to a unique limit measure $\mathrm{DH}(v)$, called the Duistermaat–Heckman measure of v.*

Proof. This can be proved by following the argument in [BC11, Cut13, LM09, KK08]. From the proof, we also see that the measure $\mathrm{DH}(v)$ is independent of the choice of r, the positive integer chosen at the beginning of this section to ensure rK_X is Cartier.

Here weak convergence of Radon measures means convergence against continuous compactly supported test functions. If Y is a point, this has been proved in [BC11], so we only consider the case when $\dim Y > 0$. Note that

$$\mathrm{DH}_m = \frac{1}{(rm)^n} \frac{d}{dt} \dim_{\mathbb{C}}(R_m / \mathcal{F}_v^{trm} R_m). \tag{2.4}$$

In the following, for simplicity of notation, we assume $r = 1$ and it follows from the proof that the limit measure does not change if we replace L with a multiple. By the compactness of Radon measures and the uniform estimates (2.20), we know that DH_m is weakly precompact. As in [BC11], by the dominated convergence theorem, it suffices to show that the limit

$$\lim_{m \to \infty} \frac{1}{m^n} \dim_{\mathbb{C}}(R_m / \mathcal{F}_v^{tm} R_m) \tag{2.5}$$

exists for almost every $t \geq 0$. Indeed, in the case $\dim Y > 0$, we can show that the above limit exists for every $t \geq 0$.

The strategy is to apply the following theorem due to [KK08] and [LM09].

Theorem 2.2 ([LM09])**.** *Denote by $\mathbb{N}$ the set of non-negative integers and let $\Gamma \subset \mathbb{N}^{n+1}$ be a sub-semigroup. For $m \in \mathbb{N}$, we denote $\Gamma_m = \Gamma \cap (\mathbb{N}^n \times \{m\})$, which can be viewed as a subset of $\mathbb{N}^n$. Suppose Γ satisfies the following three conditions:*

(1). $\Gamma_0 = \{0\}$;
(2). There exist finitely many vectors $(v_i, 1)$ spanning a semigroup $B \subset \mathbb{N}^{n+1}$ such that $\Gamma \subset B$;
(3). Γ generates $\mathbb{Z}^{n+1}$ as a group.

Then the limit

$$\lim_{m\to\infty}\frac{\#\Gamma_m}{m^n}$$

exists and equals the Euclidean volume of the Okounkov body of the semigroup Γ.

Take a log resolution $\varphi:\widetilde{X}\to X$, then the center of v on $\widetilde{X}$ satisfies that $\overline{\{c_{\widetilde{X}}(v)\}}\subset\varphi^{-1}(\pi^{-1}(p))$. Choose a point $\widetilde{q}\in\overline{\{c_{\widetilde{X}}(v)\}}$ such that $c_{\widetilde{X}}(v)$ is smooth in a neighborhood of $\tilde{q}$. In an affine chart around $\widetilde{q}$ we assume $c_{\widetilde{X}}(v)$ is cut out by k regular functions $z_1,\cdots,z_k$. Then $v(z_i)>0$ for all $i=1,\cdots,k$. We may add regular functions $z_{k+1},\cdots,z_n$ such that $\{z_1,\cdots,z_n\}$ form local analytic coordinates around $\widetilde{q}$.

Choose rationally independent positive real numbers $\alpha_1,\cdots,\alpha_n$, with

$$\alpha_i\leq\min\{v(z_i),1\},i=1,\cdots,k. \tag{2.6}$$

Consider two quasi-monomial valuations val and $\widetilde{v}$ given by

$$\mathrm{val}(f)=\min\{\sum_{i=1}^n\alpha_i\beta_i\mid c_\beta\neq 0\}.\quad \tilde{v}(f)=\min\{\sum_{i=1}^k\alpha_i\beta_i\mid c_\beta\neq 0\}, \tag{2.7}$$

for $0\neq f\in\mathcal{O}_{\widetilde{X},\widetilde{q}}$, written in the completion $\widehat{\mathcal{O}_{\widetilde{X},\widetilde{q}}}\simeq\mathbb{C}[[z_1,\ldots,z_n]]$ as $f=\sum_\beta c_\beta z^\beta$ with $c_\beta\in\mathbb{C}$. By the choice of α_i (2.6) and [JM12, Proposition 3.1], we know that for $f\in\mathcal{O}_{\widetilde{X},c_{\widetilde{X}}(v)}$,

$$v(f)\geq\tilde{v}(f). \tag{2.8}$$

Since the α_i's are rationally independent, $f\in\mathcal{O}_{\widetilde{X},\widetilde{q}}\setminus\{0\}$, we can define

$$\overrightarrow{\mathrm{val}}(f):=(\mathrm{val}_1(f),\cdots,\mathrm{val}_n(f))\in\mathbb{N}^n \tag{2.9}$$

if $\mathrm{val}(f)=\mathrm{val}_1(f)\alpha_1+\cdots+\mathrm{val}_n(f)\alpha_n$. Then we have the following result.

Lemma 2.3. *There exists a positive constant C such that for any $m\geq 0$ and $0\neq s\in R_m$, we have*

$$\sum_{i=1}^n\mathrm{val}_i(s)\leq C(\widetilde{v}(s)+m). \tag{2.10}$$

Proof. We consider the blowup $\varphi':X'\to\widetilde{X}$ of $\widetilde{X}$ along the subvariety $\overline{\{c_{\widetilde{X}}(v)\}}$ and let E denote the exceptional divisor. Then there exists a positive constant C such that for any $0\neq s\in R_m$, there exists a non-negative integer $c(s)$ satisfying

$$c(s)\leq C\widetilde{v}(s), \tag{2.11}$$

and s induces a non-zero section s' in

$$H^0(E,mL-c(s)E). \tag{2.12}$$

Note that the valuation val also naturally induces a valuation val$'$ on E. Without loss of generality, we may assume $\mathrm{val}(z_1)=\min\{\mathrm{val}(z_i):i=1,\cdots,k\}$. Let $q'\in X'$ denote the point, around which we have local analytic coordinates: $\{z_1,u_i=\frac{z_i}{z_1},z_j\mid i=2,\cdots,k,j=k+1,\cdots,n\}$, where E is locally given by $\{z_1=0\}$. Then we define val$'$ to be the quasi-monomial valuation on E centered at q' with the following weights:

$$\mathrm{val}'(u_i)=\alpha_i-\alpha_1,i=2,\cdots,k,\quad \mathrm{val}'(z_j)=\alpha_j,j=k+1,\cdots,n. \tag{2.13}$$

Then since $\alpha_1\leq 1$, we can easily check that

$$\sum_{i=1}^n\mathrm{val}_i(s)\leq\frac{\mathrm{val}(s)}{\min_{i=1}^n\alpha_i}\leq\frac{c(s)+\mathrm{val}'(s')}{\min_{i=1}^n\alpha_i}. \tag{2.14}$$

Fix a very ample line bundle A such that both $L+A$ and $A-E$ are very ample, then we have injective maps:

$$H^0(E, mL - c(s)E) \hookrightarrow H^0(E, (m + c(s))(L + 2A - E)). \tag{2.15}$$

By definition, val$'$ is a quasi-monomial valuation on E, then by [BKMS15, Proposition 2.12], there is a positive constant C independent of $s \in R_m$ and m such that

$$\mathrm{val}'(s') \leq C(c(s) + m). \tag{2.16}$$

Then (2.10) follows from (2.11), (2.14), and (2.16). □

Since the valuation val is centered at a closed point and the weights α_i are rationally independent, from (2.8) and (2.10), we obtain that there exists a positive number N, independent of m and t such that

$$\begin{aligned}\dim_{\mathbb{C}}(R_m/\mathcal{F}_v^{tm}R_m) =& \#\{(d_1, \cdots, d_n) \in \overrightarrow{\mathrm{val}}(R_m) \mid \sum_{i=1}^n d_i \leq N(t+1)m\} \\ &- \#\{(d_1, \cdots, d_n) \in \overrightarrow{\mathrm{val}}(\mathcal{F}_v^{tm}R_m) \mid \sum_{i=1}^n d_i \leq N(t+1)m\}.\end{aligned} \tag{2.17}$$

Then we have two sub-semigroups of $\mathbb{N}^{n+1}$ given by

$$\begin{aligned}\Gamma &= \{(d_1, \cdots, d_n, m) \mid (d_1, \cdots d_n) \in \overrightarrow{\mathrm{val}}(R_m), \sum_{i=1}^n d_i \leq N(t+1)m\}, \\ \Gamma' &= \{(d_1, \cdots, d_n, m) \mid (d_1, \cdots d_n) \in \overrightarrow{\mathrm{val}}(\mathcal{F}_v^{tm}R_m), \sum_{i=1}^n d_i \leq N(t+1)m\}.\end{aligned} \tag{2.18}$$

It is then clear that Γ and Γ' satisfy Condition (1) and Condition (2), since

$$\Gamma \subset \{(d_1, \cdots, d_n, m) \mid \sum_{i=1}^n d_i \leq N(t+1)m\}. \tag{2.19}$$

Therefore we have a uniform dimension estimate for $m \geq 1$,

$$\dim_{\mathbb{C}}(R_m/\mathcal{F}_v^{>\lambda m}R_m) \leq C'm^n(\lambda+1)^n, \tag{2.20}$$

which implies that DH_m is well-defined.

Since $\Gamma' \subset \Gamma$, to check Condition (3), it suffices to show it holds for Γ'. We may also allow N to increase if necessary. As L is relatively ample, we can find m_0 such that for all $m \geq m_0$, there exists an $f_m \in R_m$ with

$$\mathrm{val}(f_m) = 0. \tag{2.21}$$

Note that v induces a valuation on $\mathcal{O}_{Y,p}$ as there is a natural inclusion $\mathcal{O}_{Y,p} \hookrightarrow K(X)$ via π^*. Since $\dim Y > 0$ and v is centered in $\pi^{-1}(p)$, we can choose a non-zero element $h \in \mathfrak{m}_p \subset \mathcal{O}_{Y,p}$ such that

$$v(h) \geq t + 1. \tag{2.22}$$

Note that by the choice of h, we have $h^m R_m \subset \mathcal{F}_v^{tm}R_m$ and for $m \geq t$, we have

$$h^m R_{m+1} \subset \mathcal{F}_v^{t(m+1)}R_{m+1}. \tag{2.23}$$

Therefore by choosing m large, we get

$$(\vec{0}, 1) = (\overrightarrow{\mathrm{val}}(h^m f_{m+1}), m+1) - (\overrightarrow{\mathrm{val}}(h^m f_m), m) \tag{2.24}$$

is in the group generated by Γ' (with a possibly larger N). Again, since L is relatively ample, one can directly check that for each z_i, there are $g_i, s_i \in R_{m_i}$ such that

$$\mathrm{val}(z_i) = \mathrm{val}(g_i) - \mathrm{val}(s_i) \tag{2.25}$$

and then we consider

$$g_i' = h^{m_i} g_i, \text{ and } s_i' = h^{m_i} s_i, \tag{2.26}$$

which are both contained in $\mathcal{F}_v^{tm_i} R_{m_i}$. Then we know that

$$(e_i, 0) = (\overrightarrow{\mathrm{val}}(g_i'), m_i) - (\overrightarrow{\mathrm{val}}(s_i'), m_i) \tag{2.27}$$

is in the group generated by Γ' (with a possibly larger N), where $\{e_i\}_{i=1}^n$ is the standard basis of $\mathbb{Z}^n$. This shows that Condition (3) holds so one can apply Theorem 2.2 to get the conclusion. □

We will use the notion of *log discrepancy* $A_X(\cdot) \in [0, \infty]$ for valuations, as defined in [JM12, Section 5]. We only remark here that if X has klt singularities and v is nontrivial, then $A_X(v)$ is strictly positive. Moreover the definition of log discrepancy has been generalized to valuations on normal varieties [DFH09, BdFFU15], although here we focus on the case where X has klt singularities.

Let $(\pi : X \to Y, p)$ be a Fano fibration germ. We denote by $\mathrm{Val}^*_{\pi,p}$ the set of valuations v with $A_X(v) < \infty$ and centered in $\pi^{-1}(p)$.

Definition 2.4 (Weighted volume of a valuation)**.** *Given $v \in \mathrm{Val}^*_{\pi,p}$, we define its weighted volume by*

$$\mathbb{W}(v) := e^{A_X(v)} \int_{\mathbb{R}} e^{-x} \,\mathrm{DH}(v). \tag{2.28}$$

Remark 2.5. When we need to emphasize the dependence on the Fano fibration, we will use the notation $\mathbb{W}_\pi(v)$.

From the estimate (2.20), it is easy to see that $\mathbb{W}(v)$ is always finite and

$$\mathbb{W}(v) = e^{A_X(v)} \lim_{m\to\infty} \int_{\mathbb{R}} e^{-x} \mathrm{DH}_m(v). \tag{2.29}$$

We will be interested in minimizing $\mathbb{W}$ on the space $\mathrm{Val}^*_{\pi,p}$. This is closely related to the study of Kähler-Ricci shrinkers and Kähler-Ricci flows. We will explain more details in Section 4. The above definition incorporates several special situations studied previously:

Example 2.6. *If π is the identity map, then L is trivial and $R_m = \mathcal{O}_{X,p}$ for all $m \geq 0$. One can check that*

$$\mathbb{W}(v) = e^{A_X(v)} \,\mathrm{vol}(v), \tag{2.30}$$

where

$$\mathrm{vol}(v) := \lim_{m\to\infty} \frac{n!}{m^n} \dim_{\mathbb{C}}(\mathcal{O}_{X,p}/\mathfrak{a}_m(v)). \tag{2.31}$$

Here $\mathfrak{a}_m(v) := \{f \in \mathcal{O}_{X,p} : v(f) \geq m\}$ denotes the valuation ideal of v. This is a variant of the normalized volume defined by Li [Li18]*, but they lead to the same minimization problem. The reason is that the normalized volume is scaling invariant and to minimize $\mathbb{W}(v)$ for all valuations, it is sufficient to minimize it over the slice*

$$\{v \in \mathrm{Val}^*_{\pi,p} \mid A_X(v) = n\}. \tag{2.32}$$

The functional $\mathbb{W}$ has the additional advantage that it is not scaling invariant and it is possible to obtain the strict uniqueness of a minimizer.

Example 2.7. *If Y is a point, then X is a Fano variety. In this case, the Duistermaat–Heckman measure* DH *is not normalized to be a probability measure, and we have*

$$\mathbb{W}(v) = \frac{(-K_X)^n}{n!} e^{\widetilde{\beta}(v)} \tag{2.33}$$

where $\widetilde{\beta}$ is the functional introduced in [HL24]. *Note that on a fixed X, the factor $(-K_X)^n$ is not important, however, we expect that the weighted volume functional $\mathbb{W}$ provides a more natural quantity in studying boundedness questions; see the discussion in Section 4.3.*

Definition 2.8. *We define the weighted volume of*

- *a Fano fibration germ $(\pi : X \to Y, p)$ to be $\mathbb{W}(\pi, p) := \inf_{v \in \mathrm{Val}^*_{\pi,p}} \mathbb{W}(v)$;*
- *a Fano variety X to be $\mathbb{W}(X) := \mathbb{W}(\pi : X \to \{pt\}, \{pt\})$;*
- *a klt singularity (X, q) to be $\mathbb{W}(X, q) := \mathbb{W}(\mathrm{Id} : X \to X, q)$.*

Proof of Theorem 1.3. Given a Fano fibration germ $(\pi : X \to Y, p)$ and $q \in \pi^{-1}(p)$. Let $v \in \mathrm{Val}^*_{\mathrm{Id},q} \subset \mathrm{Val}^*_{\pi,p}$ be a valuation centered at q with finite log discrepancy. Let R_m be the stalk of $\pi_*\mathcal{O}(mL)$ at p. Then by taking a local trivialization near q, we obtain embeddings of vector spaces

$$R_m/\mathcal{F}^\lambda_v R_m \hookrightarrow \mathcal{O}_{X,q}/\mathcal{F}^\lambda_v \mathcal{O}_{X,q}. \tag{2.34}$$

Then by (2.4) and Tonelli theorem, we have

$$\int_{[0,\infty)} e^{-x} \,\mathrm{DH}_m(v)(dx) = \frac{1}{(rm)^n} \int_0^\infty e^{-t} \dim_{\mathbb{C}}\big(R_m/\mathcal{F}^{trm}_v R_m\big)\, dt.$$

Then by the definition (2.4) and (2.29), we obtain that

$$\mathbb{W}_\pi(v) \leq \mathbb{W}_{\mathrm{Id}}(v). \tag{2.35}$$

Taking the infimum over $v \in \mathrm{Val}^*_{\mathrm{Id},q}$, we obtain

$$\mathbb{W}(\pi, p) \leq \mathbb{W}(X, q). \tag{2.36}$$

□

As mentioned in the introduction, Theorem 1.3 can also be viewed as an algebro-geometric analogue of the Bishop–Gromov volume comparison. Moreover it can also be viewed as an analogue of the monotonicity of the μ-functional [LW23] for Kähler-Ricci shrinkers discussed in Section 4.1. For Kähler-Einstein metrics, the algebro-geometric analogue of the Bishop–Gromov volume comparison was established by [Liu18]. The differential geometric heuristics suggest that the equality in (1.3) holds if and only if π is an isomorphism of germs.

3. Futaki invariant and K-stability of polarized Fano fibrations

In this section, we formulate a notion of K-stability for general polarized Fano fibrations. A preliminary attempt was given in [CDS24, Section 7] and one can see that the objects considered in [CDS24, Conjecture 7.5] are naturally polarized Fano fibrations.

Let $(\pi : X \to Y, \xi)$ be a polarized Fano fibration. In the following discussion, for the simplicity of notation, we assume K_X is a Cartier divisor, and the general case for K_X being $\mathbb{Q}$-Cartier can be addressed in a similar manner.

3.1. **Weighted volume of Reeb vector fields.** By definition a polarized Fano fibration $(\pi : X \to Y, \xi)$ naturally defines a Fano fibration germ with a center at the vertex o of Y. Any element η in the Reeb cone induces a valuation in $\mathrm{Val}^*_{\pi,o}$. Then we can define the weighted volume of η as the weighted volume of this valuation induced by η. Moreover, in the following, we explain that this weighted volume is determined by the weights of η acting on global sections of $-mK_X$.

For a given Reeb vector field η, there are two associated filtrations on the section ring

$$\mathcal{R} := \bigoplus_{m\geq 0} R_m := \bigoplus_{m\geq 0} H^0(X, -mK_X).$$

Firstly, the $\mathbb{T}$-action admits a canonical lifting to $-K_X$ (as an automorphism of X can push forward tangent vectors) and hence acts on the section ring $\mathcal{R}$. Then we have a filtration on $\mathcal{R}$ induced by the weight decomposition $R_m = \bigoplus_{\alpha\in \mathrm{Lie}(\mathbb{T})^*} R_{m,\alpha}$:

$$\mathrm{wt}_\eta(s) := \min\{\langle \alpha, \eta\rangle \mid s = \sum_\alpha s_\alpha, s_\alpha \neq 0\} \text{ and } \mathcal{F}^\lambda_{\mathrm{wt}_\eta} R_m := \{s \in R_m \mid \mathrm{wt}_\eta(s) \geq \lambda\}. \tag{3.1}$$

Secondly, η induces a valuation v_η on the function field $K(X)$ as follows. Choose a nonempty $\mathbb{T}$-invariant affine open subset $U \subset X$ and consider the weight decomposition $\mathcal{O}_X(U) = \bigoplus_\alpha \mathcal{O}_X(U)_\alpha$. For $0 \neq f \in \mathcal{O}_X(U)$, we set

$$v_\eta(f) := \min\{\langle \alpha, \eta\rangle \mid f = \sum_\alpha f_\alpha,\ f_\alpha \in \mathcal{O}_X(U)_\alpha,\ f_\alpha \neq 0\}. \tag{3.2}$$

The minimum-weight rule is multiplicative, so it extends uniquely to $K(X) = \mathrm{Frac}\, \mathcal{O}_X(U)$ by

$$v_\eta(f/g) = v_\eta(f) - v_\eta(g), \qquad 0 \neq f, g \in \mathcal{O}_X(U),$$

and we set $v_\eta(0) = +\infty$. It is straightforward to show that this definition is independent of U. As η is a Reeb vector field, the restriction of v_η to $K(Y)$ is centered at o. By the properness of π, the valuation v_η has a center in $\pi^{-1}(o)$. As before, by locally trivializing the bundle $-K_X$, we obtain a valuation along with the associated filtration on $\mathcal{R}$. The following result is the local discrepancy calculation in [LX18, Lemma 2.18].

Lemma 3.1. *For $s \in R_m$, we have*

$$v_\eta(s) = \mathrm{wt}_\eta(s) + mA_X(v_\eta). \tag{3.3}$$

In particular, the filtration associated with v_η and wt_η on R_m differs only by a translation. We define the measure

$$\mathrm{DH}(\eta) := \iota_{A_X(v_\eta)*}(\mathrm{DH}(v_\eta)) \tag{3.4}$$

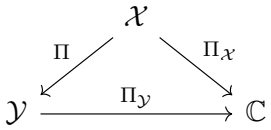
where $\iota_{A_X(v_\eta)}(x) = x - A_X(v_\eta)$ denotes the translation on $\mathbb{R}$. Then we have

$$\mathbb{W}(\eta) := \mathbb{W}(v_\eta) = \int_{\mathbb{R}} e^{-x}\mathrm{DH}(\eta), \tag{3.5}$$

which by (3.3) can be computed using the weights of wt_η on $\mathcal{R}$.

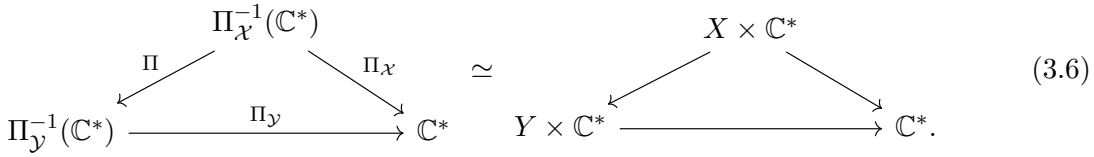
3.2. **K-stability of polarized Fano fibrations.** We first define *special test configurations* for a polarized Fano fibration.

Definition 3.2 (Special test configurations). *Let $(\pi : X \to Y, \xi)$ be a polarized Fano fibration. A special test configuration $(\Pi : \mathcal{X} \to \mathcal{Y}, \xi, \eta)$ of $(\pi : X \to Y, \xi)$ consists of a commutative diagram, where both $\Pi_{\mathcal{X}}$ and $\Pi_{\mathcal{Y}}$ are surjective flat morphisms,*



satisfying the following conditions:

(1). η is a holomorphic vector field on $\mathcal{X}$ generating a $\mathbb{C}^*$-action on $\mathcal{X}$ and $\mathcal{Y}$ such that Π, $\Pi_{\mathcal{X}}$ and $\Pi_{\mathcal{Y}}$ are $\mathbb{C}^*$-equivariant, where the base $\mathbb{C}$ is equipped with its standard $\mathbb{C}^*$-action. Moreover there is a $\mathbb{C}^*$-equivariant isomorphism



(3.6)

(2). Under the isomorphism (3.6), the vector field ξ extends to a holomorphic vector field on $\mathcal{X}$, still denoted by ξ, which commutes with η. Moreover, ξ generates a torus action on both $\mathcal{X}$ and $\mathcal{Y}$ with respect to which Π is equivariant.

(3). $\mathcal{X}$ is $\mathbb{Q}$-Gorenstein and $(\Pi_0 : \mathcal{X}_0 \to \mathcal{Y}_0, \xi)$ is a polarized Fano fibration, where $\mathcal{X}_0$ and $\mathcal{Y}_0$ denote the central fibers of $\Pi_{\mathcal{X}}$ and $\Pi_{\mathcal{Y}}$ respectively.

Definition 3.3 (Product test configuration). *A special test configuration is said to be a product test configuration, if there is a ξ-equivariant isomorphism $\mathcal{X} \simeq X \times \mathbb{C}$ and $\eta = \eta_0 + z\partial_z$, where η_0 is a holomorphic vector field on X commuting with ξ and z is the standard coordinate on $\mathbb{C}$.*

Next, we define the *Futaki invariant* of a special test configuration.

Definition 3.4 (Futaki invariant). *For a special test configuration $(\Pi : \mathcal{X} \to \mathcal{Y}, \xi, \eta)$ with central fiber $(\Pi_0 : \mathcal{X}_0 \to \mathcal{Y}_0, \xi)$, we define*

$$\mathrm{Fut}_{\xi}(\eta) := -\frac{d}{dt}\Big|_{t=0} \mathbb{W}_{\Pi_0}(\xi + t\eta). \tag{3.7}$$

Definition 3.5 (K-stability). *A polarized Fano fibration $(\pi : X \to Y, \xi)$ is said to be K-semistable, if for any special test configuration, the Futaki invariant $\mathrm{Fut}_{\xi}(\eta) \geq 0$.*

A polarized Fano fibration $(\pi : X \to Y, \xi)$ is said to be K-polystable if it is K-semistable and a special test configuration has vanishing Futaki invariant if and only if it is a product test configuration.

Remark 3.6. This unifies the notion of K-stability with respect to special test configurations for a polarized Fano variety (when Y is a point) and a Fano cone (when π is the identity map). As in [CS18, BHJ17, BLXZ23], it is possible to define K-stability for more general test configurations, which is expected to be equivalent to the above definition using only special test configurations. In the case when Y is a point this was proved by [LX14] and [HL23]. In the case where π is the identity map, there has been progress [LW24]. Moreover, many other related notions, like the δ-invariant [FO18, BLXZ23] and Ding-stability, can be generalized to polarized Fano fibration setting.

The following lemma works for any polarized Fano fibration $(\pi : X \to Y, \xi)$, and ensures that the Futaki invariant in (3.7) is well-defined.

Lemma 3.7. *For a polarized Fano fibration $(\pi : X \to Y, \xi)$ and $0 \neq \eta \in \mathrm{Lie}(\mathbb{T})$, we have*

$$\left.\frac{d}{dt}\right|_{t=0} \mathbb{W}(\xi + t\eta) = -\lim_{m\to\infty} \frac{1}{m^n} \sum_{\alpha \in \mathrm{Lie}(\mathbb{T})^*} e^{-\langle \frac{\alpha}{m}, \xi\rangle} \langle \frac{\alpha}{m}, \eta\rangle \dim_{\mathbb{C}} R_{m,\alpha}, \tag{3.8}$$

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \mathbb{W}(\xi + t\eta) = \lim_{m\to\infty} \frac{1}{m^n} \sum_{\alpha \in \mathrm{Lie}(\mathbb{T})^*} e^{-\langle \frac{\alpha}{m}, \xi\rangle} \langle \frac{\alpha}{m}, \eta\rangle^2 \dim_{\mathbb{C}} R_{m,\alpha} > 0. \tag{3.9}$$

Proof. When $\dim Y = 0$ this is proved by [BWN14] and it also follows from a more general result in [BLXZ23] using Okounkov bodies. In the following, we assume $\dim Y > 0$ and we use Okounkov bodies as in [BLXZ23]. We only deal with (3.8) and a similar argument works for (3.9). Let $\Lambda_m = \{\alpha \in \mathrm{Lie}(\mathbb{T})^* \mid R_{m,\alpha} \neq 0\}$. In particular, Λ_0 is the weight lattice of the $\mathbb{T}$-action on Y.

Since $-K_X$ is relatively ample and ξ is a polarization, there exist positive constants A, B independent of $m \geq 0$, such that for $\alpha \in \Lambda_m$, we have

$$\langle \alpha, \xi\rangle \geq -Am, \text{ and } |\langle \alpha, \eta\rangle| \leq A\langle \alpha, \xi\rangle + Bm. \tag{3.10}$$

Moreover, similar to (2.20), we have the following uniform dimension upper bound for $\lambda \geq 0$ and $m \geq 1$:

$$\sum_{\langle \alpha, \xi\rangle \leq \lambda m} \dim_{\mathbb{C}}(R_{m,\alpha}) \leq C(\lambda + 1)^n m^n. \tag{3.11}$$

We have a sequence of Radon measures on $\mathbb{R}^2$ given by

$$\mathrm{DH}_m(\xi, \eta) := \frac{1}{m^n} \sum_{\alpha \in \Lambda_m} \dim_{\mathbb{C}}(R_{m,\alpha}) \delta_{\left(\langle \frac{\alpha}{m}, \xi\rangle, \langle \frac{\alpha}{m}, \eta\rangle\right)}. \tag{3.12}$$

Consider functions $H_m : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$H_m(x, y) = \frac{1}{m^n} \dim_{\mathbb{C}} \left(\mathcal{F}^{ym}_{\mathrm{wt}_\eta} R_m / \mathcal{F}^{xm}_{\mathrm{wt}_\xi} R_m \cap \mathcal{F}^{ym}_{\mathrm{wt}_\eta} R_m \right). \tag{3.13}$$

Then we obtain $\mathrm{DH}_m(\xi, \eta) = -\frac{\partial^2}{\partial x \partial y} H_m$ in the sense of distributions. Using (3.11) and some elementary arguments, to prove (3.8), it suffices to show that the right-hand side limit in (3.8) exists. It is clear that the limit depends linearly on η, so we may assume that there exists an $\alpha_\eta \in \Lambda_0$ such that $\langle \alpha_\eta, \eta\rangle > 0$, i.e., there exists $f_\eta \in R_{0,\alpha_\eta}$ such that

$$\mathrm{wt}_\eta(f_\eta) > 0. \tag{3.14}$$

As in the proof of Proposition 2.1, we again appeal to Theorem 2.2. One can extend wt_ξ to be a rank n valuation $\overrightarrow{\mathrm{val}}$ and obtain

$$H_m(x, y) = \frac{\#\Gamma_m - \#\Gamma'_m}{m^n}, \tag{3.15}$$

where Γ and Γ' are two sub-semigroups of $\mathbb{N}^{n+1}$ given by

$$\begin{aligned}
\Gamma &= \{(d_1, \cdots, d_n, m) \mid (d_1, \cdots, d_n) \in \overrightarrow{\mathrm{val}}(\mathcal{F}^{ym}_{\mathrm{wt}_\eta} R_m \cap \bigoplus_{\langle \alpha, \xi\rangle \leq Nm(|x|+1)} R_{m,\alpha})\}, \\
\Gamma' &= \{(d_1, \cdots, d_n, m) \mid (d_1, \cdots, d_n) \in \overrightarrow{\mathrm{val}}(\mathcal{F}^{xm}_{\mathrm{wt}_\xi} R_m \cap \mathcal{F}^{ym}_{\mathrm{wt}_\eta} R_m \cap \bigoplus_{\langle \alpha, \xi\rangle \leq Nm(|x|+1)} R_{m,\alpha})\}.
\end{aligned} \tag{3.16}$$

It is clear that they satisfy Conditions (1) and (2) of Theorem 2.2. Using f_η, one can choose $N \gg 1$, and argue as in (2.21)–(2.27) to see that Condition (3) also holds. Then as in the proof of Proposition 2.1, the dominated convergence theorem implies the measure $\mathrm{DH}_m(\xi,\eta)$ converges weakly to a unique Radon measure $\mathrm{DH}(\xi,\eta)$ on $\mathbb{R}^2$. Moreover, by (3.10) the support of $\mathrm{DH}(\xi,\eta)$ is contained in the region

$$\{(x,y) \mid x \geq -A \text{ and } |y| \leq Ax + B\}.$$

Then by (3.11), one can easily show that the right-hand side limit in (3.8) exists and equals

$$-\int_{\mathbb{R}^2} ye^{-x}\mathrm{DH}(\xi,\eta).$$

Lastly, we verify that the second derivative in (3.9) is always positive. Indeed since η is nonzero, up to replacing η by $-\eta$, we can find $n+1$ algebraically independent sections $s_i \in R_{m_i,\alpha_i}$ such that

$$\langle \frac{\alpha_i}{m}, \eta\rangle > 0. \tag{3.17}$$

Using these sections, it is easy to see the right-hand side of (3.9) is positive. □

It follows from (3.9) that the weighted volume is a strictly convex function on the Reeb cone. In particular, there exists at most one critical (and minimum) point of $\mathbb{W}$ on the Reeb cone. This fact generalizes the volume minimization principle in [MSY08] for Ricci-flat Kähler cones, in [TZ02] for compact Kähler–Ricci shrinkers and in [CDS24] for non-compact Kähler–Ricci shrinkers with bounded Ricci curvature.

Remark 3.8. Following [BWN14, BLXZ23, BLQ24], two-dimensional Duistermaat–Heckman measures can be defined for general Fano fibrations, associated with a pair of filtrations satisfying certain "properness" conditions. This approach would play a role in studying the convexity of various functionals and the uniqueness of the weighted volume minimizer.

3.3. **The case when X is smooth.** The above definition of weighted volume is motivated to give an algebro-geometric interpretation of the weighted volume in [CDS24] which was defined in terms of symplectic geometry. Suppose $(\pi : X \to Y, \xi)$ is a polarized Fano fibration with X smooth. We now explain that the two definitions of weighted volume coincide.

First, using the fact that the polarized Fano fibration can be equivariantly embedded into $\mathbb{C}^{N_1} \times \mathbb{CP}^{N_2}$, it is easy to construct a $\mathbb{T}$-invariant positively curved smooth hermitian metric on $-K_X$ with curvature form $-\sqrt{-1}\omega$ (then $\omega \in 2\pi c_1(X)$), such that the $\mathbb{T}$-action is Hamiltonian with respect to ω with a moment map $\mu : X \to \mathrm{Lie}(\mathbb{T})^*$ and $u_\xi := \langle \mu, \xi\rangle$ is proper and bounded below on X. Note that moment maps are not unique and they differ by an element in $\mathrm{Lie}(\mathbb{T})^*$: a choice of the moment map corresponds to a lift of the $\mathbb{T}$-action to the line bundle $-K_X$. Here we have a canonical lifting for the $\mathbb{T}$-action, as automorphisms of the manifold can push forward tangent vectors. In the following, we work with the moment map associated with this canonical lifting and note that the properness of u_ξ is independent of the choice of the moment map.

In [CDS24], the *analytic* weighted volume is defined as

$$\mathbb{W}^{an}(\eta) := \frac{1}{(2\pi)^n}\int_X e^{-\langle \mu,\eta\rangle}\frac{\omega^n}{n!}. \tag{3.18}$$

Using the Duistermaat–Heckman formula [PW94], it is shown in [CDS24] that (3.18) is well-defined for η belonging to an open convex cone $\Lambda \subset \mathrm{Lie}(\mathbb{T})$, which contains ξ.

Proposition 3.9. *In the above setting we have*

$$\mathbb{W}^{an}(\xi) = \mathbb{W}(\xi),$$

where $\mathbb{W}(\xi)$ *is defined in* (3.5).

Proof. Since $\pi : X \to Y$ is a polarized Fano fibration, X is weakly pseudo-convex, and $-K_X$ admits a smooth hermitian metric with positive curvature form. Then by [Dem97, Chapter VIII-Theorem 5.6], we know that for all $k \geq 1$ and $m \geq 0$

$$H^k(X, -mK_X) = 0.$$

Then we can apply [CDS24, Proposition 7.2], which uses Wu's Riemann-Roch type formula [Wu03], to obtain that

$$\mathbb{W}^{an}(\xi) = \lim_{m\to\infty} \frac{1}{m^n} \mathrm{char} H^0(X, -mK_X)\left(\frac{\xi}{m}\right), \tag{3.19}$$

where char denotes the (formal) character for the $\mathbb{T}$-action. Here by the choice of the moment map, the right-hand side of (3.19) equals

$$\lim_{m\to\infty} \frac{1}{m^n} \sum_{\alpha\in \mathrm{Lie}(\mathbb{T})^*} \dim_{\mathbb{C}}(R_{m,\alpha}) e^{-\langle \frac{\alpha}{m}, \xi\rangle}. \tag{3.20}$$

By Proposition 2.1 and Lemma 3.1, the measure

$$\frac{1}{m^n} \sum_{\alpha\in \mathrm{Lie}(\mathbb{T})^*} \dim_{\mathbb{C}}(R_{m,\alpha}) \delta_{\langle \frac{\alpha}{m}, \xi\rangle} \tag{3.21}$$

converges weakly to $\mathrm{DH}(\xi)$. Therefore, by the definition of $\mathbb{W}(\xi)$, it is sufficient to verify the validity of interchange of limits and integration. Integrating by parts, we obtain that

$$\frac{1}{m^n} \sum_{\langle \frac{\alpha}{m}, \xi\rangle \geq A} \dim_{\mathbb{C}}(R_{m,\alpha}) e^{-\langle \frac{\alpha}{m}, \xi\rangle} \leq \frac{1}{m^n} \int_A^\infty e^{-t} \dim_{\mathbb{C}}(R_m / \mathcal{F}^{>tm}_{\mathrm{wt}_\xi} R_m) dt. \tag{3.22}$$

Then the conclusion follows from (3.22) and the uniform dimension estimate (2.20). □

As in [CDS24], there is an explicit expression for the moment map corresponding to the canonical lifting. By construction there is a $\mathbb{T}$-invariant Ricci potential h for ω, i.e.,

$$\mathrm{Ric}(\omega) + \sqrt{-1}\partial\bar{\partial} h = \omega. \tag{3.23}$$

(Note that it is proved in [CDS24, Lemma 5.10] that if (ω, f) is a Kähler-Ricci shrinker, the moment map always exists and f is automatically invariant under the $\mathbb{T}$-action. So in this case we can choose $h = f$.)

Then we can normalize the moment map for $\eta \in \mathrm{Lie}(\mathbb{T})$ as follows:

$$\Delta_\omega u_\eta + u_\eta + \frac{1}{2} J\eta \cdot h = 0. \tag{3.24}$$

We can easily check that this normalization corresponds to the canonical lifting of the $\mathbb{T}$-action. Indeed, note that

$$\Delta_\omega u_\eta = -\frac{1}{2}\mathrm{div}(J\eta) = \frac{1}{2}\frac{\mathcal{L}_{-J\eta}\omega^n}{\omega^n}. \tag{3.25}$$

Therefore at a zero x of η, by (3.24) and (3.25), for $s \in (-K_X)_x$ we have

$$\mathcal{L}_\eta s = \sqrt{-1} u_\eta s. \tag{3.26}$$

This implies that u_η is the moment map corresponding to the canonical lifting of the $\mathbb{T}$-action on $-K_X$.

The weighted volume of a shrinker is closely related to Perelman's $\mathcal{W}$-entropy, μ-functional and reduced volume [Per02]. Recall that on a compact real m-dimensional Riemannian manifold (X, g) with a smooth function ϕ and $\tau > 0$, Perelman's $\mathcal{W}$-entropy and μ-functional are defined as

$$\mathcal{W}(g, \phi, \tau) = \int_X \left(\tau\left(|\nabla\phi|^2 + R\right) + \phi - m\right) \frac{e^{-\phi}}{(4\pi\tau)^{m/2}} dV_g, \tag{3.27}$$

$$\mu(g, \tau) = \inf\{\mathcal{W}(g, \phi, \tau) \mid \int_X \frac{e^{-\phi}}{(4\pi\tau)^{m/2}} dV_g = 1\}. \tag{3.28}$$

This definition can be generalized to non-compact Ricci shrinkers [LW23]. In this paper, we work with Kähler-Ricci shrinkers (g, J, ω, f) under the convention (1.1) and normalize f such that (1.2) holds, that is we have

$$\Delta_\omega f + f - |\nabla^{1,0} f|^2 = 0. \tag{3.29}$$

The results from [LW23, CN09] show that the function $\mu(g, \tau)$ is decreasing on $(0, \frac{1}{2})$ and increasing on $(\frac{1}{2}, \infty)$ and moreover,

$$\frac{1}{(2\pi)^n} \int_X e^{-(f+n)} \frac{\omega^n}{n!} = e^{\mu(g, \frac{1}{2})}. \tag{3.30}$$

Let ξ denote the soliton vector field as before. Then by Proposition 3.9 and (3.18), we obtain that on a Kähler-Ricci shrinker, the algebraically defined weighted volume satisfies

$$\mathbb{W}(\xi) = e^{\mu(g, \frac{1}{2}) + n}. \tag{3.31}$$

4. The conjectures

4.1. **YTD type conjecture.** It is natural to study singular Kähler–Ricci shrinkers. We propose the following definition.

Definition 4.1. *A singular Kähler–Ricci shrinker consists of the data (X, ξ, ω, f) satisfying the following conditions:*

- *(X, ξ) admits a polarized Fano fibration structure;*
- *ω is a ξ-invariant positive $(1,1)$-current with locally bounded potentials, whose restriction to X^{reg} is a smooth Kähler form;*
- *f is a continuous function on X, smooth on the regular locus X^{reg}. Moreover, f is proper and bounded from below, and satisfies $\xi = J\nabla f$ on X^{reg};*
- *$\int_X |\nabla f|^k e^{-f} \omega^n < \infty$ for every $k \geq 0$;*
- *on X^{reg}, the soliton equation holds:*

$$\mathrm{Ric}(\omega) + \sqrt{-1}\partial\bar{\partial} f = \omega. \tag{4.1}$$

When Y is a point, this definition reduces to that of a Kähler–Ricci shrinker on a polarized Fano variety. When π is the identity map, we expect it to reduce to that of a Ricci-flat Kähler cone metric on a Fano cone (see Conjecture 4.4). When X is smooth, a singular Kähler–Ricci shrinker as defined above is smooth on X and is known to define a complete metric. Thus, in this case, our definition agrees with the standard definition of a Kähler–Ricci shrinker on a smooth complex manifold. Throughout this section, a Kähler–Ricci shrinker is understood in the sense of Definition 4.1, allowing the underlying variety to be singular.

Given a polarized Fano fibration $(\pi : X \to Y, \xi)$, we say that it admits a Kähler–Ricci shrinker if there exists a singular Kähler–Ricci shrinker (X, ξ, ω, f) on X satisfying $J\nabla f = \xi$. This leads to the following YTD conjecture for singular polarized Fano fibrations.

Conjecture 4.2. *A polarized Fano fibration $(\pi : X \to Y, \xi)$ admits a Kähler–Ricci shrinker, unique up to the action of* $\mathrm{Aut}(X, \xi)$*, if and only if it is K-polystable.*

It is also an open question to understand the metric geometry of a Kähler-Ricci shrinker. For example, one expects suitable conical behavior at infinity.

Conjecture 4.3 (Asymptotic conicity)**.** *A Kähler-Ricci shrinker is weakly asymptotically conical in the sense that it has a unique asymptotic cone at infinity, which is a Kähler cone metric on Y.*

The interest of these conjectures lies in the analysis in the non-compact setting, *without* a priori prescribing the behavior at infinity. This is in the spirit of a free boundary problem. Some explicit examples of Kähler-Ricci shrinkers using the Calabi ansatz have been constructed in [Koi90, Cao96, FIK03, DW11, FW11, Li10].

The uniqueness part of Conjecture 1.2 was proved in [Cif22] in the toric setting. Under the assumption of quadratic curvature decay, Conjecture 4.3 holds by [MW17, CDS24], and there is work by Esparza [Esp25] on the uniqueness part of Conjecture 1.2. In the toric setting, one expects that a toric Fano fibration always admits a unique K-polystable polarization and hence admits a toric Kähler-Ricci shrinker. This is known in the compact setting [WZ04, BB13] and progress has been made by [CCD24a] in the non-compact setting.

When π is the identity map, by definition a Ricci-flat Kähler cone metric is a Kähler-Ricci shrinker. On the other hand, the K-stability notions for these two objects coincide on a Fano cone. The uniqueness part in Conjecture 4.2 would imply the converse:

Conjecture 4.4. *On a Fano cone X, every Kähler-Ricci shrinker is a Ricci-flat Kähler cone metric.*

This is a purely differential geometric statement. We give heuristic arguments here under extra assumptions. Let g denote the Riemannian metric of a Kähler-Ricci shrinker on X with soliton vector field ξ. Suppose one can apply the results in [DS17] to get a 2-step degeneration to its unique tangent cone $(\mathcal{C}, g_{\mathcal{C}})$ at the vertex $o \in X$, which is a Ricci-flat Kähler cone. Then by the monotonicity [LW20] and upper-semicontinuity [CCG$^+$07, Lemma 6.28] of the μ-functional, we can get

$$\mu(g_{\mathcal{C}}, \frac{1}{2}) \geq \limsup_{\tau \to 0} \mu(\tau^{-1} g, \frac{1}{2}) = \limsup_{\tau \to 0} \mu(g, \tau) \geq \mu(g, \frac{1}{2}). \tag{4.2}$$

Then using [CN09] (see also [LW20, Proposition 3]), one sees that (see (3.31) for the smooth case)

$$e^{\mu(g_{\mathcal{C}}, \frac{1}{2}) + n} = \mathbb{W}(v_g) \quad \text{and} \quad e^{\mu(g, \frac{1}{2}) + n} = \mathbb{W}(\xi), \tag{4.3}$$

where v_g denotes the valuation induced by the metric g [DS17, Section 3.2]. Since $g_{\mathcal{C}}$ is a Ricci-flat Kähler cone metric, the valuation v_g minimizes the weighted volume [LX18], and then the equality holds in (4.2). From the variation of the μ-functional, we can see that g is a cone metric with Reeb vector field ξ, implying g must be Ricci-flat since its tangent cone at o is Ricci-flat.

4.2. **Weighted volume minimization.** We propose the following 2-step degeneration conjecture for weighted volume minimization. The goal is to degenerate a Fano fibration germ to a K-polystable polarized Fano fibration.

Conjecture 4.5. *(2-step degeneration for weighted volume) Let $(\pi : X \to Y, p)$ be a Fano fibration germ.*

(1). *There is a unique minimizer* v_* *of* $\mathbb{W}$ *on* $\mathrm{Val}^*_{\pi,p}$*, which is quasi-monomial.*
(2). *The graded ring* $\mathrm{Gr}_{v_*}(\mathcal{O}_{Y,p})$ *is finitely generated and* $\bigoplus_{m\geq 0}\mathrm{Gr}_{v_*}(R_m)$ *is a finitely generated algebra over* $\mathrm{Gr}_{v_*}(\mathcal{O}_{Y,p})$*.*
(3). *Let* $W=\mathrm{Spec}(\mathrm{Gr}_{v_*}(\mathcal{O}_{Y,p}))$ *and* $Z=\mathrm{Proj}_W(\bigoplus_{m\geq 0}\mathrm{Gr}_{v_*}(R_m))$*. Then* v_* *induces a vector field* ξ *on* Z *which descends to* W*. The natural map* $\tilde{\pi}: Z\to W$ *and* ξ *define a K-semistable polarized Fano fibration. Moreover,* v_* *is uniquely characterized by this K-semistability property.*
(4). *There is an equivariant test configuration degenerating* $(\tilde{\pi}: Z\to W,\xi)$ *to a unique K-polystable polarized Fano fibration* $(\widehat{\pi}:\mathcal{S}\to\mathcal{C},\xi)$*.*

This brings together a few special situations studied previously. When π is the identity map this essentially reduces to the minimization of normalized volume for klt singularities in [Li18]. When Y is a point, this reduces to the 2-step degeneration theory for Fano varieties [DS20, HL24]. In these two cases the conjecture is known by the work of many people; see [XZ22, BLXZ23] and the references therein. One expects that suitable extensions of these techniques can be used to prove Conjecture 4.5.

It is also interesting to investigate more closely the set of weighted volumes of Fano fibration germs. As usual one expects that the weighted volume is lower semi-continuous in flat families of Fano fibration germs, which would imply the maximal volume of a birational Fano fibration germ in dimension n is uniquely achieved by the standard $(\mathrm{Id}:\mathbb{C}^n\to\mathbb{C}^n,0)$. Compare [Fuj18, LX19, BL21] for results on similar questions in related settings. Here, a Fano fibration germ $(\pi: X\to Y,p)$ is called *birational* if π is birational. Moreover, as an analogue of the ODP volume gap conjecture (see [SS17]), it is natural to ask:

Question 4.6. *Determine the second largest weighted volume of a birational Fano fibration germ in dimension* n*.*

There are two natural candidates: $\pi:\mathrm{Bl}_0(\mathbb{C}^n)\to\mathbb{C}^n$ and the ODP klt singularity (with $\pi=\mathrm{Id}$). Note that there is a well-known FIK Kähler-Ricci shrinker [FIK03] on $\pi:\mathrm{Bl}_0(\mathbb{C}^n)\to\mathbb{C}^n$, whose Reeb vector field is a multiple of the standard rotational vector field on $\mathbb{C}^n$ and whose weighted volume has an explicit formula; see [CDS24, Appendix A.4].

4.3. **Moduli theory.** The study of canonical metrics in Kähler geometry is closely related to the construction of moduli spaces in algebraic geometry. One prominent example is the case of Kähler-Einstein metrics on Fano varieties. Building on the differential geometric results of [DS14, CDS15, SSY16], the compactified moduli space of smooth K-polystable Fano varieties was constructed by [LWX19] and [Oda15]. Later this has been generalized to non-smoothable Fano varieties in algebraic geometry by [Jia20, BLX22, XZ20, LXZ22]. There are also related studies of moduli theory for Fano cone singularities (corresponding to Ricci-flat Kähler cone metrics) and for polarized Fano varieties (corresponding to compact Kähler-Ricci shrinkers) in both differential and algebraic geometry; see for example [Sun23, HLW21, XZ24, Oda24a, Oda24b, Che24]. We expect all of these can be incorporated into one theory:

Conjecture 4.7 (K-moduli for polarized Fano fibrations)**.** *Given a positive integer* n *and* $\epsilon>0$*, there exists a projective moduli space parametrizing K-polystable polarized Fano fibrations of dimension* n *and weighted volume at least* ϵ*.*

Here the weighted volume of a K-polystable polarized Fano fibration $(\pi: X\to Y,\xi)$ is defined to be $\mathbb{W}(\xi)$ (cf. (3.5)). In particular, we expect:

- (Boundedness) The set of K-semistable polarized Fano fibrations with weighted volume bounded below by a positive constant $\epsilon>0$ forms a bounded family.

This question is related to the weak compactness of Ricci shrinkers (see [HLW21, LLW21] for instance). Notice that in the case when Y is a point, the boundedness question above is known, as a consequence of (1.3); for instance, one way to see this is through the finite degree formula in [XZ21] and the boundedness result in [HMX14].

In birational geometry, a well-known question is on the termination of flips in the minimal model program. This is related to bounding the weighted volume from below for the Fano fibrations coming from the extremal contractions along a fixed minimal model program.

4.4. **Kähler-Ricci flow I: finite time singularities.** Consider the Ricci-flow on a compact Kähler manifold (X, ω)

$$\frac{\partial}{\partial t}\omega(t) = -\mathrm{Ric}(\omega(t)) \tag{4.4}$$

with $\omega(0) = \omega$. In [ST17], it is proposed that analyzing this flow constitutes an analytic analogue of the minimal model program (with scaling) in birational geometry, often referred to as the analytic minimal model program. It is well-known [TZ06] that the flow exists exactly on $[0, T)$, where T can be numerically computed as the supremum of all $t > 0$ such that $[\omega] - 2\pi t c_1(X)$ is a Kähler class.

If K_X is not nef, then $T < \infty$ and the flow must encounter finite time singularities at T. From now on we assume X is projective and the initial Kähler class $[\omega(0)] = 2\pi c_1(L)$ for some line bundle L. Then by the Kawamata basepoint free theorem, we obtain a Fano fibration $\pi : X \to Y$ induced by the semi-ample $\mathbb{Q}$-line bundle $L+TK_X$. When X is Fano and $[\omega] \in 2\pi c_1(X)$, it is shown in [CSW18] that there is a 2-step degeneration from the normalized Kähler-Ricci flow to a unique Kähler-Ricci shrinker on a $\mathbb{Q}$-Fano variety. Motivated by this we make the following.

Motivated by this we make the following

Conjecture 4.8 (2-step degeneration for Kähler-Ricci flows)**.** *Let $\pi : X \to Y$ be the Fano fibration at the singular time $T < \infty$. For any $p \in Y$ and $q \in \pi^{-1}(p)$,*

- *The Kähler-Ricci flow $\omega(t)$ naturally induces a weighted volume minimizing (hence K-semistable) valuation on $(\pi : X \to Y, p)$.*
- *The tangent flow of $\omega(t)$ at q is unique, and is given by the Kähler-Ricci shrinker $(\widehat{\pi} : \mathcal{S} \to \mathcal{C}, \xi)$ canonically associated with the Fano fibration germ $(\pi : X \to Y, p)$ via Conjecture 4.5 and Conjecture 4.2.*

Remark 4.9. As mentioned in the introduction, one can more generally consider a Kähler-Ricci flow space-time with singularities in a suitable sense and study the tangent flow at a singularity. This will then include the above situation as well as the case of tangent cones at singularities of Kähler-Einstein metrics in the sense of [DS17] (which are self-similar Ricci flow solutions).

Remark 4.10. When the Fano fibration π is birational, Conjecture 4.8 gives a precise characterization of the Kähler-Ricci shrinker appearing in the shrinker-to-expander transition, conjectured in [Son13] for the analytic minimal model program. As in [Son13], one expects there exists a Kähler-Ricci expander $\mathcal{E}$, whose asymptotic cone coincides with that of the Kähler-Ricci shrinker produced in this conjecture. The underlying variety of $\mathcal{E}$ is the unique canonical model for $K_{\mathcal{S}}$ over $\mathcal{C}$, whose existence is guaranteed by [BCHM10]. There has been some work constructing Kähler-Ricci expanders; see [CD20] and the references therein.

Theorem 1.1 is a first step towards proving Conjecture 4.8. The uniqueness of tangent flows with singularities is in general a difficult question in geometric analysis, and the above

conjecture leads to an algebro-geometric way of proving such uniqueness. This strategy has been successfully realized in [DS17] for the uniqueness of tangent cones of singular Kähler-Einstein metrics and in [CSW18] for the uniqueness of tangent flows for the Ricci flow on Fano manifolds. It is possible to apply the techniques in [DS17, CSW18] and the weak compactness theory in [Bam20a] to tackle Conjecture 4.8. Some partial progress towards generalizing the infinitesimal result of [DS17] has been made by [Hal23a]. We mention that there are also results explicitly determining the tangent flows in special settings; see [CCD24b, JST23].

Regarding the first part of the conjecture, we propose the following construction. Write $R_m = (\pi_* \mathcal{O}(-mK_X))_p$ for $m \geq 0$. Fix $q \in \pi^{-1}(p)$ and then for any sequence of times $T_i \nearrow T$, we can pass to a subsequence to obtain convergence in $C^\infty_{\mathrm{loc}}(X \times [0,T)$ to a positive solution

$$u_t := \lim_{i\to\infty} K(q, T_i; \cdot, \cdot) : X \times [0,T) \to \mathbb{R}$$

of the conjugate heat equation [MM15, Lemma 2.2]. Let

$$d\mu_t = u_t \frac{\omega(t)^n}{n!}, \qquad h_t = u_t h_{\omega(t)},$$

where $h_{\omega(t)}$ is the induced hermitian metric on $-K_X$. Thus the volume form associated with h_t is the probability measure $d\mu_t$. For $0 \neq s \in R_m$ represented on a neighborhood U of $\pi^{-1}(p)$, choose a smooth cutoff $0 \leq \chi \leq 1$ supported compactly in U and equal to 1 near $\pi^{-1}(p)$, and set

$$\|s\|^2_{q,t} := \int_U \chi |s|^2_{h_t^{\otimes m}} \, d\mu_t.$$

We expect that the asymptotic degree

$$d_q(s) := \lim_{t\to T^-} \frac{\log \|s\|^2_{q,t}}{\log(T-t)} \tag{4.5}$$

exists as a finite real number and is independent of U and χ. These degrees may be negative. Set $d_q(0) = +\infty$. We further expect that

$$\begin{aligned} d_q(s_1 s_2) &= d_q(s_1) + d_q(s_2), \\ d_q(s_1 + s_2) &\geq \min\{d_q(s_1), d_q(s_2)\}, \end{aligned} \tag{4.6}$$

with the last inequality concerning sections of the same tensor degree. Assuming these assertions, we obtain a valuation on the rational function field $K(X)$ by

$$v_q(s_1/s_2) := d_q(s_1) - d_q(s_2), \qquad 0 \neq s_1, s_2 \in R_m,$$

and $v_q(0) = +\infty$.

For the self-similar flow generated by a Kähler–Ricci shrinker with soliton vector field ξ, take $d\mu_t$ to be the soliton conjugate heat measure. The scaling of the canonically lifted action gives $d_q(s) = \mathrm{wt}_\xi(s)$ for algebraic sections with finite weighted norms, and hence $v_q = v_\xi$.

For the general flow, we expect that $v_q \in \mathrm{Val}^*_{\pi,p}$ and v_q is independent of $q \in \pi^{-1}(p)$ and of the choice of limiting conjugate heat kernel u_t, and is the weighted volume minimizing valuation in Conjecture 4.5.

4.5. **Kähler–Ricci flow II: ancient solutions.** By drawing an analogy with the theory of complete Ricci-flat Kähler metrics with Euclidean volume growth, one can also reverse the rescaling direction and consider non-collapsed ancient solutions to the Kähler-Ricci flow. By definition this means a Kähler-Ricci flow $g(t)(t \in (-\infty, 0])$ of complete Kähler metrics on a fixed complex manifold X with Perelman's μ-functional $\mu(g(t), -t)$ uniformly bounded from below. Algebro-geometrically we introduce:

Definition 4.11 (Non-collapsing weak Fano fibration)**.** *A non-collapsing weak Fano fibration is a fibration $\pi : X \to Y$, where X has klt singularities, Y is a normal affine variety and $-K_X$ is a relatively nef and big $\mathbb{Q}$-Cartier divisor.*

Notice that the relative bigness of $-K_X$ and the affineness of Y reflect the essence of the non-collapsing condition. Denote $\mathfrak{R}_m = H^0(X, -mK_X)$. In particular, $\mathfrak{R}_0$ is the ring of regular functions on X (and on Y), which is a finitely generated $\mathbb{C}$-algebra. Let $\mathfrak{R} := \bigoplus_{m\geq 0} \mathfrak{R}_m$ be the global anti-canonical ring. By the relative base point free theorem [KM98, Theorem 3.24], $\mathfrak{R}$ is a finitely generated graded algebra over $\mathfrak{R}_0$. A negative valuation on a non-collapsing weak Fano fibration $\pi : X \to Y$ consists of a function $d : \mathfrak{R} \setminus \{0\} \to \mathbb{R}$ satisfying

- $d(s_1 + s_2) \geq \min\{d(s_1), d(s_2)\}$;
- $d(s_1 s_2) = d(s_1) + d(s_2)$;
- $d(f) \leq 0$ for $0 \neq f \in \mathfrak{R}_0$.

Definition 4.12. *We say a negative valuation is K-polystable if the associated graded algebra is finitely generated and with the reversed grading it defines a K-polystable polarized Fano fibration $(\pi : \mathcal{S} \to \mathcal{C}, \xi)$.*

We make the following conjecture, which serves as a global version of Conjecture 4.8.

Conjecture 4.13 (Tangent flows at infinity)**.** *Let $(X, \omega(t))(t \in (-\infty, 0])$ be a non-collapsed ancient Kähler-Ricci flow. Then*

- *X admits a non-collapsing weak Fano fibration structure and the flow naturally induces a K-polystable negative valuation;*
- *The tangent flow at infinity of $\omega(t)$ is unique and is given by the Kähler-Ricci shrinker determined by the K-polystable negative valuation via Conjecture 4.2.*

As in Section 4.4, we propose a formula for the negative valuation induced by the flow. For this construction, assume that the curvature is bounded on each compact time interval. Fix $q \in X$ and let

$$d\mu_t = u_t \frac{\omega(t)^n}{n!}, \qquad h_t = u_t h_{\omega(t)},$$

where u_t is the conjugate heat kernel with pole at the regular space-time point $(q, 0)$; see [Bam20b, Section 2.3 and Definition 2.4], with the noncompact extension justified in [Bam25, Appendix A]. Assuming the conjectural algebraic structure on X, for $0 \neq s \in \mathfrak{R}_m$ we set

$$\|s\|^2_{q,t} := \int_X |s|^2_{h_t^{\otimes m}} \, d\mu_t, \qquad d(s) := -\lim_{t\to-\infty} \frac{\log \|s\|^2_{q,t}}{\log(-t)}. \tag{4.7}$$

We expect that algebraic sections have finite norms, that these limits exist as finite real numbers independently of q, and that they define a K-polystable negative valuation.

For the self-similar model, using the soliton conjugate heat measure, an algebraic section $s = \sum_\alpha s_\alpha$ satisfies

$$d(s) = -\max\{\langle \alpha, \xi\rangle : s_\alpha \neq 0\}.$$

This explains the reversal of the filtration grading. For example, for the Gaussian shrinker on $\mathbb{C}^n$, writing $e = \partial_{z_1} \wedge \cdots \wedge \partial_{z_n}$, we have $d(z^\alpha e^{\otimes m}) = mn - |\alpha|$.

Note that a "no-semistability at infinity" phenomenon is expected in Conjecture 4.13, which is motivated by [SZ23]. Moreover, conjecturally non-collapsed ancient Kähler-Ricci flows can be characterized by the following algebraic data:

- A choice of a K-polystable polarized Fano fibration $(\pi : \mathcal{S} \to \mathcal{C}, \xi)$.

- A choice of a K-polystable negative valuation on a non-collapsing weak Fano fibration $(\pi : X \to Y)$, which degenerates it to $(\pi : \mathcal{S} \to \mathcal{C}, \xi)$.
- A choice of a Kähler class on X (corresponding to $[\omega(0)]$).

Remark 4.14. The weighted volume of the Kähler-Ricci shrinkers obtained in Conjecture 4.8 and 4.13 is related to Perelman's $\mathcal{W}$-entropy along the Ricci flow; see [MM15, Xu17, MZ21, Hal23b].

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